The 8th "STARS of MATHEMATICS" Competition – Seniors
November 29, 2014 ★★★ ICHB – Bucharest

Problem 1. Prove that for any integer $n > 1$ there exist infinitely many pairs (x, y) of integers $1 < x < y$, such that $x^n + y \mid x + y^n$.

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Solution. Let us then try to find solutions with $y = kx$, where $k > 1$ is integer. We then need $x^{n-1} + k \mid 1 + k^n x^{n-1}$.

When n is odd we can take $x^{n-1} + k \mid (x^{n-1})^n + k^n$, and so $x^{n-1} + k \mid 1 + k^n x^{n-1} = 1 + x^{n-1}\left((x^{n-1})^n + k^n\right) - x^{n^2-1}$, thus $x^{n-1} + k \mid x^{n^2-1} - 1$. We see $(x, y) = (m, m(m^{n^2-1} - m^{n-1} - 1))$ will thus be solutions for any integer $m > 1$.

When n is even we can take $x^{n-1} + k \mid (x^{n-1})^n - k^n$, and so $x^{n-1} + k \mid 1 + k^n x^{n-1} = 1 - x^{n-1}\left((x^{n-1})^n - k^n\right) + x^{n^2-1}$, thus $x^{n-1} + k \mid x^{n^2-1} + 1$. We see $(x, y) = (m, m(m^{n^2-1} - m^{n-1} + 1))$ will thus be solutions for any integer $m > 1$.

We may in fact unify these above results, by claiming $\boxed{(x, y) = (m, m(m^{n^2-1} - m^{n-1} + (-1)^n))}$ will be solutions for any integer $m > 1$.

Alternatively, looking at $P(y) = y^n + x$ as some polynomial $P \in \mathbb{Z}[y]$, by the Euclidean division algorithm we have

$$P(y) = (y + x^n)Q(y) + P(-x^n).$$

It is then enough to take $x > 1$ and

$$y = |P(-x^n)| - x^n = x^{n^2} - x^n + (-1)^n x.$$

Even more complex polynomials may be similarly done. ■

Remarks. For just the value $n = 3$ the problem has been also asked in the Juniors Section.

Problem 2. Let N be an arbitrary positive integer. Prove that if, from among any n consecutive integers larger than N, one may select 7 of them, pairwise co-prime, then $n \geq 22$.

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Solution. A model (that shows the value $n = 21$ is too small) may be exhibited by taking $a = 210k + 199$, $k \in \mathbb{N}$, and the set $A = \{a+1, a+2, \ldots, a+21\}$. Then we may take $A_2 = \{a+1, a+3, \ldots, a+21\}$, $A_3 = \{a+2, a+8, a+14, a+20\}$, $A_5 = \{a+6, a+16\}$, $A_7 = \{a+4, a+18\}$, $A' = \{a+10\}$, $A'' = \{a+12\}$, such that the elements in A_p are divisible by p, for $p \in \{2, 3, 5, 7\}$. Since these 6 subsets of above will make up a partition of A, it means, by the pigeonhole principle, that for any selection of 7 elements from A there will be 2 belonging to a same A_p, hence not co-prime.

The idea behind all this is to try to bunch together as many multiples of 2, 3, 5 and/or 7 (the smallest primes) as possible, in order for the pigeonhole principle to readily be applied. After playing around a bit, we should come up with sets $A_2, A_3, A_5, A_7, A', A''$; now it is left to find an a such that the elements in A_p are divisible by p for $p \in \{2, 3, 5, 7\}$, i.e. to solve the system of congruences $a \equiv -1 \pmod 2$, $a \equiv -2 \pmod 3$, $a \equiv -6 \pmod 5$, $a \equiv -4 \pmod 7$. The least such positive solution (warranted to exist by CRT – the Chinese Remainder Theorem) is $a = 199$, gotten without too large an effort. We may rewrite $a \equiv -1 \pmod{10}$, so $a = 10b - 1$, then $10b - 1 \equiv 1 \pmod 3$, or $b \equiv 2 \pmod 3$, so $b = 3c + 2$, finally $30c + 19 \equiv -4 \pmod 7$, or $c \equiv -1 \pmod 7$, so $c = 7d - 1$, and so $a = 210d - 11 = 210k + 199$, $k \in \mathbb{N}$. The model obtained is almost unique in its simplicity. ■

Remarks. Notice that the model for $n = 21$ is too high to just be "guessed"; it rather needs to be inferred via some deductive reasoning. It may be proved that in fact for $n = 22$ one may always make such a selection, thus value 22 **is** the threshold value. However, the reasoning is a little tiresome, and based on some amount of casework, so it is not quite recommending itself for being asked.

Problem 3. Let positive integers M, m, n be such that $1 \leq m \leq n$, $1 \leq M \leq \dfrac{m(m+1)}{2}$, and let $A \subseteq \{1, 2, \ldots, n\}$ with $|A| = m$. Prove there exists a subset $B \subseteq A$ with

$$0 \leq \sum_{b \in B} b - M \leq n - m.$$

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Solution. Let $A = \{a_1, a_2, \ldots, a_m\}$, where

$$a_0 = 0 < 1 \leq a_1 < a_2 < \cdots < a_m \leq n.$$

Since $a_k \geq k$ for all $1 \leq k \leq m$, it follows that

$$a_1 + a_2 + \cdots + a_m \geq \frac{m(m+1)}{2} \geq M,$$

therefore the family of the subsets $B = \{a_{k_1}, a_{k_2}, \ldots, a_{k_\ell}\}$ that have $\sum_{b \in B} b \geq M$ is non-empty. Consider now such a subset B with $\sum_{b \in B} b - M$ minimal. Then $0 \leq \sum_{b \in B} b - M \leq a_{k_1} - a_{k_1 - 1} - 1$, by dint of the minimality of B, since otherwise the new set $B' = (B \cup \{a_{k_1 - 1}\}) \setminus \{0, a_{k_1}\}$ has the property

$$0 \leq \sum_{b' \in B'} b' - M < \sum_{b \in B} b - M.$$

On the other hand, $a_{k_1} - a_{k_1 - 1} - 1 \leq n - m$, from the fact that the distance between two elements of $A \cup \{a_0\}$ bearing consecutive indices is at most $n - m + 1$, since

$$n \geq a_m = \sum_{\ell = 1}^{m} (a_\ell - a_{\ell - 1}) \geq (m - 1) + (a_k - a_{k-1})$$

for any $1 \leq k \leq m$. ■

Remarks. The right side inequality $\sum_{b \in B} b - M \leq n - m$ may easily become an equality – for example when $M = 1$ and $A = \{n - m + 1, n - m + 2, \ldots, n\}$, $B = \{n - m + 1\}$. Other such cases may occur; see below.

The problem might have been formulated for just the value $M = \dfrac{m(m+1)}{2}$, but that is a little **unfair**, as it puts some emphasis on this particular value of M, which is in fact irrelevant to the proof. Then the inequality on the right side $\sum_{b \in B} b - \dfrac{m(m+1)}{2} \leq n - m$ may become an equality – for $m = n$ (obvious, forcing $B = A = \{1, 2, \ldots, n\}$), or for $m = 1$ (if taking $B = A = \{n\}$). Other such cases may occur, such as for example $n = 4$, $m = 3$, with $A = \{1, 3, 4\}$, $B = \{3, 4\}$.

This problem turns to be a (major) improvement on some older (and weaker) attempt (from a different author), at some previous Viitori-Olimpici competition.

Problem 4. At the junction of some countably infinite number of roads sits a greyhound. On one of the roads a hare runs, away from the junction. The only thing known is that the (maximal) speed of the hare is strictly less than the (maximal) speed of the greyhound (but not their precise ratio). Does the greyhound have a strategy for catching the hare in a finite amount of time?

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Solution. Let us start by making several notations towards describing the steps $\Phi_1, \Phi_2, \ldots, \Phi_n, \ldots$ of a strategy Φ, which within a finite amount of time will allow the greyhound to catch the hare. Take as origin the junction point, $g_1 = 0$ and $h_0 = d > 0$ (the distances of the greyhound, respectively hare, from origin at the moment in time $T_1 = 0$). Denote the speed of the greyhound by g (thus the speed h of the hare is so that $h < g$, but $g - h$ is unknown).

Consider a function $r : \mathbb{N}^* \to \mathbb{N}^*$ such that the fibre $r^{-1}(k)$ is infinite for each $k \in \mathbb{N}^*$. Then, for any K, each fibre $r^{-1}(k)$ will contain numbers larger than K. Such a function r is for example $r(n) = \omega(n)$, the arithmetic function counting the number of distinct primes in the factorization of n (with just redefining $r(1) = 1$). We will now describe step Φ_n.

• Step Φ_n applies at the moment in time T_n.

• Compute $h_n = n + g\left(1 - \frac{1}{n}\right) T_n$, the largest distance from origin the hare could have achieved at the moment in time T_n, for $d \le n$ and $h \le g\left(1 - \frac{1}{n}\right)$.

• Compute $t_n = \dfrac{g_n + h_n}{g/n}$, the largest time needed by the greyhound to run back to origin, then pick the road $r(n)$ and catch the hare (if it was on this road) under the conditions of the above.

• Finally define $T_{n+1} = T_n + t_n$, and $g_{n+1} = g t_n - g_n$ (the distance of the greyhound from origin at the end of step Φ_n).

All it is left to do is notice that the steps of the strategy are correctly defined, and since the number n increases by 1 with each step of the strategy, at some moment in time it will become large enough so that simultaneously we get to have $n \ge d$ and $n \ge \dfrac{g}{g-h}$, which is equivalent to $h \le g\left(1 - \frac{1}{n}\right)$, and also that $r(n)$ is the very road on which the hare runs, hence the hare will be caught. ■

Remarks. The same question was asked, for only a couple of roads, in the Juniors Section.

A similarly flavoured question – but simpler in all extents (only two roads; **known** ratio of speeds) – has been asked a few years ago in the Russian Olympiad.

> From a police station situated on a straight road, infinite in both directions, a thief has stolen a police car. Its maximal speed equals 90% of the maximal speed of a police cruiser. When the theft is discovered some time later, a policeman engages to pursue the thief on a cruiser. However, he does not know in which direction along the road the thief has gone, nor does he know how long ago the car has been stolen. Is it possible for the policeman to catch the thief?

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