The 8th "STARS of MATHEMATICS" Competition – Juniors
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Problem 1. Prove there exist infinitely many pairs (x, y) of integers $1 < x < y$, such that $x^3 + y \mid x + y^3$.

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Solution. Let us then try to find solutions with $y = kx$, where $k > 1$ is integer. We then need $x^2 + k \mid 1 + k^3x^2 = 1 + x^2((x^2)^3 + k^3) - x^8$, thus $x^2 + k \mid x^8 - 1$. We therefore see $\boxed{(x, y) = (m, m(m^8 - m^2 - 1))}$ will be solutions, and this for any integer $m > 1$.

Alternatively, looking at $P(y) = y^3 + x$ as some polynomial $P \in \mathbb{Z}[y]$, by the Euclidean division algorithm we have

$$P(y) = (y + x^3)(y^2 - yx^3 + x^6) - (x^9 - x).$$

It is then enough to take $x > 1$ and

$$y = (x^9 - x) - x^3 = x(x^8 - x^2 - 1).$$

Thus in fact we find (at least) a solution for each $x > 1$. ■

Remarks. For all exponents $n > 1$ (rather than just $n = 3$) the question has been also asked in the Seniors Section.

Problem 2. Determine all integers $n \geq 1$ for which the numbers $1, 2, \ldots, n$ may be ordered as $a_1, a_2, \ldots, a_n$, in such a way that the average $\dfrac{a_1 + a_2 + \cdots + a_k}{k}$ is an integer for all values $1 \leq k \leq n$.

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Solution. We claim it can only be done for $\boxed{n \in \{1, 3\}}$, with $(a_1, a_2, a_3) = (1, 3, 2)$ when $n = 3$. It clearly cannot be done for $n = 2$, so assume in the sequel that $n > 3$.

Denote $s_k = a_1 + a_2 + \cdots + a_k$; then n must divide the sum $s_n = \dfrac{n(n+1)}{2}$, forcing n to be odd. Now $n - 1$ must divide the sum $s_{n-1} = s_n - a_n = \dfrac{n(n+1)}{2} - a_n$, therefore we must have $2(n-1) \mid n(n+1) - 2a_n$, so $n - 1 \mid 2(a_n - 1)$. There are only three possibilities

- $a_n = 1$, but then $2(n-1) \nmid n(n+1) - 2a_n = (n+2)(n-1)$, since $n + 2$ is odd;
- $a_n = n$, but then $2(n-1) \nmid n(n+1) - 2a_n = n(n-1)$, since n is odd;
- $a_n = \dfrac{n+1}{2}$, and so $2(n-1) \mid n(n+1) - 2a_n = (n+1)(n-1)$, since $n + 1$ is even. Now $n - 2$ must in turn divide the sum $s_{n-2} = s_{n-1} - a_{n-1} = \dfrac{(n+1)(n-1)}{2} - a_{n-1}$, therefore we must have $2(n-2) \mid (n+1)(n-1) - 2a_{n-1}$, so $n - 2 \mid 2a_{n-1} - 3$. There exists only one possibility, namely $a_{n-1} = \dfrac{n+1}{2}$, but that is unavailable, since the value is already used by a_n. ■

Remarks. It is however possible for the (infinitely many) numbers $1, 2, \ldots, n, \ldots$ to be ordered as $a_1, a_2, \ldots, a_n, \ldots$, in such a way that the sum $a_1 + a_2 + \cdots + a_k$ is divisible by k for all integers $1 \leq k$. This is a quite classical result, and I encourage those not knowing it already to strive to prove it. The proof involves an elegant use of the Chinese Remainder Theorem, as opposed to some cumbersome application of a greedy algorithm.

Problem 3.

i) Show there exist (not necessarily distinct) non-negative real numbers $a_1, a_2, \ldots, a_{10}$; $b_1, b_2, \ldots, b_{10}$, with $a_k + b_k \leq 4$ for all $1 \leq k \leq 10$, such that $\max\{|a_i - a_j|, |b_i - b_j|\} \geq 4/3 > 1$ for all $1 \leq i < j \leq 10$.

ii) Prove for any (not necessarily distinct) non-negative real numbers $a_1, a_2, \ldots, a_{11}$; $b_1, b_2, \ldots, b_{11}$, with $a_k + b_k \leq 4$ for all $1 \leq k \leq 11$, there exist $1 \leq i < j \leq 11$ such that $\max\{|a_i - a_j|, |b_i - b_j|\} \leq 1$.

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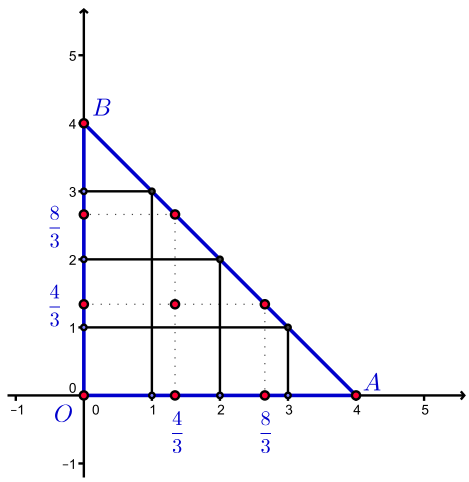

Figure courtesy of ANDREI ECKSTEIN.

Solution. Consider the pairs (a_k, b_k) as coordinates for points $p_k(a_k, b_k)$ inside (or on the border) of $\triangle OAB$ of vertices $O(0,0)$, $A(4,0)$, $B(0,4)$; notice now that we have $\max\{|a_i - a_j|, |b_i - b_j|\} = \|p_i - p_j\|_\infty$, the "distance" between p_i and p_j in the metric determined by the *maximum* norm $\|(x, y)\|_\infty := \max\{|x|, |y|\}$ in the space $\mathbb{R}^2$. This allows for a clearer visualization of the ideas in the proof.

i) Just consider the 10 points (marked in **red**) $O(0,0)$ and $(4/3, 0), (0, 4/3), (8/3, 0), (4/3, 4/3), (0, 8/3), A(4, 0), (8/3, 4/3), (4/3, 8/3), B(0, 4)$. The least "distance" between any two of them is $4/3 > 1$. The model presented is in fact unique.

ii) Consider the 10 regions into which the vertical lines $x = 1, x = 2, x = 3$ and the horizontal lines $y = 1, y = 2, y = 3$ partition the triangle OAB. By the pigeonhole principle, two of the 11 points will lie inside (or on the border) of a same region, and as such, the "distance" between them will be at most 1 (in fact a total of 15 points (the vertices of the above regions) may such be taken). ■

Remarks. Similar problems may be construed instead with $\|(x, y)\|_1 := |x| + |y|$, the *taxicab* (or *Manhattan*) norm. The knowledge of these equivalent – to the Euclidean $\|(x, y)\|_2 := \sqrt{x^2 + y^2}$ – norms is important and interesting in itself, and I urge you to read the pertinent textbook(s).

Problem 4. At a point on the real line sits a greyhound. On one of the sides a hare runs, away from the hound. The only thing known is that the (maximal) speed of the hare is strictly less than the (maximal) speed of the greyhound (but not their precise ratio). Does the greyhound have a strategy for catching the hare in a finite amount of time?

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Solution. Let us start by making several notations towards describing the steps $\Phi_1, \Phi_2, \ldots, \Phi_n, \ldots$ of a strategy Φ, which within a finite amount of time will allow the greyhound to catch the hare. For this, we will introduce some notations and concepts.

Take as origin the point where the greyhound initially sits, $g_1 = 0$ and $h_0 = d > 0$ (the distances from the origin of the greyhound, respectively hare, at the moment in time $T_1 = 0$). Denote the speed of the greyhound by g (thus the speed h of the hare is so that $0 \le h < g$, but $h/g < 1$ is unknown). Denote the two rays emanating from the origin as roads $r(1)$ and $r(2)$.

Consider the function $r\colon \mathbb{N}^* \to \{1,2\}$ given by $r(n) = 1$ if $n-1$ is even and $r(n) = 2$ if $n-1$ is odd, thus the fibre $r^{-1}(k)$ is infinite for each $1 \le k \le 2$. We will now describe step Φ_n.

• Step Φ_n applies at the moment in time T_n.

• Compute $h_n = n + g\left(1 - \dfrac{1}{n}\right) T_n$, the largest distance from origin the hare could have achieved at the moment in time T_n, for $d \le n$ and $h \le g\left(1 - \dfrac{1}{n}\right)$.

• Compute $t_n = \dfrac{g_n + h_n}{g/n}$, the largest time needed by the greyhound to run back to origin, then pick the road $r(n)$ and catch the hare (if it was on this road) under the conditions of the above.

• Finally define $T_{n+1} = T_n + t_n$, and $g_{n+1} = g t_n - g_n$ (the distance of the greyhound from origin at the end of step Φ_n).

All it is left to do is notice that the steps of the strategy are correctly defined, and since the number n increases by 1 with each step of the strategy, at some moment in time it will become large enough so that simultaneously we get to have $n \ge d$ and $n \ge \dfrac{g}{g-h}$, which is equivalent to $h \le g\left(1 - \dfrac{1}{n}\right)$, and also that $r(n)$ is the very road on which the hare runs, hence the hare will be caught. ■

Remarks. The same question was asked, for a countably infinite number of roads, in the Seniors Section.

A similarly flavoured question – but simpler in all extents (only two roads; **known** ratio of speeds) – has been asked a few years ago in the Russian Olympiad.

> From a police station situated on a straight road, infinite in both directions, a thief has stolen a police car. Its maximal speed equals 90% of the maximal speed of a police cruiser. When the theft is discovered some time later, a policeman engages to pursue the thief on a cruiser. However, he does not know in which direction along the road the thief has gone, nor does he know how long ago the car has been stolen. Is it possible for the policeman to catch the thief?

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